On the largest-eigenvalue process for generalized Wishart random matrices

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Abstract. Using a change-of-measure argument, we prove an equality in law between the process of largest eigenvalues in a generalized Wishart random-matrix process and a last-passage percolation process. This equality in law was conjectured by Borodin and Péché (2008).

1. Introduction

The past decade has witnessed a surge of interest in connections between random matrices on the one hand and applications to growth models, queueing systems, and last-passage percolation models on the other hand; standard references are Baryshnikov (2001) and Johansson (2000). In this note we prove a result of this kind: an equality in law between a process of largest eigenvalues for a family of Wishart random matrices and a process of directed last-passage percolation times.

To formulate the main result, we construct two infinite arrays of random variables on an underlying measurable space, along with a family $\{P^{\pi,\hat{\pi}}\}$ of probability measures parametrized by a positive N-vector π and a nonnegative sequence $\{\hat{\pi}_n : n \geq 1\}$. The elements of the first array $\{A_{ij} : 1 \leq i \leq N, j \geq 1\}$ are independent and A_{ij} has a complex zero-mean Gaussian distribution with variance $1/(\pi_i + \hat{\pi}_j)$ under $P^{\pi,\hat{\pi}}$. That is, both the real and complex part of A_{ij} have zero mean and variance $1/(2\pi_i+2\hat{\pi}_j)$. Write $A(n)$ for the $N\times n$ matrix formed by the first n columns of A, and define the matrix-valued stochastic process $\{M(n) : n \geq 0\}$ by setting $M(n) = A(n)A(n)^*$ for $n \geq 1$ and by letting $M(0)$ be the $N \times N$ zero matrix. We call $\{M(n) : n \geq 0\}$ a generalized Wishart random-matrix process, since the marginals have a Wishart distribution if π and $\hat{\pi}$ are identically one and zero, respectively.

The elements of the second array $\{W_{ij} : 1 \leq i \leq N, j \geq 1\}$ are independent and W_{ij} is exponentially distributed with parameter $\pi_i + \hat{\pi}_j$ under $P^{\pi,\hat{\pi}}$. We define

$$Y(N,n) = \max_{P\in\Pi(N,n)} \sum_{(ij)\in P} W_{ij},$$

where $\Pi(N,n)$ is the set of up-right paths from $(1,1)$ to (N,n). The quantity $Y(N,n)$ arises in last-passage percolation models as well as in series Jackson networks in queueing theory, see for instance Dieker and Warren (2008) or Johansson (2009).

The following theorem, a process-level equality in law between the largest eigenvalue of $M(n)$ and $Y(N,n)$, is the main result of this note. Given a matrix C, we write $\mathrm{sp}(C)$ for its vector of eigenvalues, ordered decreasingly.

Theorem 1.1. *For any strictly positive vector π and any nonnegative sequence $\hat{\pi}$, the processes $\{\mathrm{sp}(M(n))_1 : n \geq 1\}$ and $\{Y(N,n) : n \geq 1\}$ have the same distribution under $P^{\pi,\hat{\pi}}$.*

It is known from Defosseux (2008); Forrester and Rains (2006) that this holds in the 'standard' case, i.e., under the measure $P := P^{(1,\dots,1),(0,0,\dots)}$. In its stated generality, the theorem was conjectured by Borodin and Péché (2008), who prove that the laws of $Y(N,n)$ and the largest eigenvalue of $M(n)$ coincide for fixed $n \geq 1$. Our proof is based on a change-of-measure argument, which is potentially useful to prove related equalities in law.

Throughout, we use the following notation. We let $\mathbf{H}_{N,N}$ be the space of all $N \times N$ Hermitian matrices, and W^N the set $\{x \in \mathbb{R}^N : x_1 \geq \ldots \geq x_N\}$. For $x, x' \in W^N$, we write $x \prec x'$ to mean that x and x' interlace in the sense that

$$x'_1 \geq x_1 \geq x'_2 \geq x_2 \geq \ldots \geq x'_N \geq x_N.$$

2. Preliminaries

This section provides some background on generalized Wishart random matrices, and introduces a Markov chain which plays an important role in the proof of Theorem 1.1.

2.1. *The generalized Wishart random-matrix process.* Under $P^{\pi,\hat{\pi}}$, the generalized Wishart process $\{M(n) : n \geq 0\}$ from the introduction has independent increments since, for $m \geq 1$,

$$M(m) = M(m-1) + \left(A_{im}\bar{A}_{jm}\right)_{1\leq i,j\leq N}, \tag{2.1}$$

where $\bar{A}_{jm}$ is the complex conjugate of A_{jm}. In particular, the matrix-valued increment has unit rank. The matrix $M(m) - M(m-1)$ can be parameterized by its diagonal elements together with the complex arguments of A_{im} for $1 \leq i \leq N$; under $P^{\pi,\hat{\pi}}$, these are independent and the former have exponential distributions while the latter have uniform distributions on $[0, 2\pi]$. (This fact is widely used in the Box-Muller method for computer generation of random variables with a normal distribution.) Since the i-th diagonal element has an exponential distribution under $P^{\pi,\hat{\pi}}$ with parameter $\pi_i + \hat{\pi}_m$, we obtain the following proposition.

Proposition 2.1. *For any $m \geq 1$, the $P^{\pi,\hat{\pi}}$-law of $M(m) - M(m-1)$ is absolutely continuous with respect to the P-law of $M(m) - M(m-1)$, and the Radon-Nikodym derivative is*

$$\prod_{i=1}^{N}(\pi_i + \hat{\pi}_m) \exp\left(-\sum_{i=1}^{N}(\pi_i + \hat{\pi}_m - 1)(M_{ii}(m) - M_{ii}(m-1))\right).$$

2.2. *A Markov transition kernel.* We next introduce a time-inhomogeneous Markov transition kernel on W^N. We shall prove in Section 3 that this kernel describes the eigenvalue-process of the generalized Wishart random-matrix process of the previous subsection.

In the standard case ($\pi \equiv 1$, $\hat{\pi} \equiv 0$), it follows from unitary invariance (see Defosseux (2008, Sec. 5) or Forrester and Rains (2006)) that the process $\{\mathrm{sp}(M(n) : n \geq 0\}$ is a homogeneous Markov chain. Its one-step transition kernel $Q(z, \cdot)$ is the law of $\mathrm{sp}(\mathrm{diag}(z) + G)$, where $G = \{g_i \bar{g}_j : 1 \leq i, j \leq N\}$ is a rank one matrix determined by an N-vector g of standard complex Gaussian random variables. For z in the interior of W^N, $Q(z, \cdot)$ is absolutely continuous with respect to Lebesgue measure on W^N and can be written explicitly as in Defosseux (2008, Prop. 4.8):

$$Q(z, dz') = \frac{\Delta(z')}{\Delta(z)} e^{-\sum_k (z'_k - z_k)} 1_{\{z \prec z'\}} dz',$$

where $\Delta(z) := \prod_{1 \leq i < j \leq N} (z_i - z_j)$ is the Vandermonde determinant.

We use the Markov kernel Q to define the aforementioned time-inhomogeneous Markov kernels, which arise from the generalized Wishart random-matrix process. For general π and $\hat{\pi}$, we define the inhomogeneous transition probabilities $Q^{\pi,\hat{\pi}}_{n-1,n}$ via

$$Q^{\pi,\hat{\pi}}_{n-1,n}(z, dz') = \prod_{i=1}^{N} (\pi_i + \hat{\pi}_n) \frac{h_\pi(z')}{h_\pi(z)} e^{-(\hat{\pi}_n - 1) \sum_{i=1}^{N} (z'_i - z_i)} Q(z, dz'),$$

with

$$h_\pi(z) = \frac{\det\{e^{-\pi_i z_j}\}}{\Delta(\pi)\Delta(z)}. \tag{2.2}$$

Note that $h_\pi(z)$ extends to a continuous function on $(0, \infty)^N \times W^N$ (this can immediately be seen as a consequence of the Harish-Chandra-Itzykson-Zuber formula, see (3.2) below).

One can verify that the $Q^{\pi,\hat{\pi}}$ are true Markov kernels by writing $1_{\{z \prec z'\}} = \det\{1_{\{z_i < z'_j\}}\}$ and applying the Cauchy-Binet formula

$$\int_{W^N} \det\{\xi_i(z_j)\} \det\{\psi_j(z_i)\} dz = \det\left\{\int_{\mathbb{R}} \xi_i(z)\psi_j(z) dz\right\}.$$

3. The generalized Wishart eigenvalue-process

In this section, we determine the law of the eigenvalue-process of generalized Wishart random-matrix process. Although it is not essential to the proof of Theorem 1.1, we formulate our results in a setting where $\mathrm{sp}(M(0))$ is allowed to be nonzero.

Write m_μ for the 'uniform distribution' on the set $\{M \in \mathbf{H}_{N,N} : \mathrm{sp}(M) = \mu\}$. That is, m_μ is the unique probability measure invariant under conjugation by unitary matrices, or equivalently m_μ is the law of $U \,\mathrm{diag}(\mu) U^*$ where U is unitary and distributed according to (normalized) Haar measure. We define measures $P^{\pi,\hat{\pi}}_\mu$ by letting the $P^{\pi,\hat{\pi}}_\mu$-law of $\{M(n) - M(0) : n \geq 0\}$ be equal to the $P^{\pi,\hat{\pi}}$-law of $\{M(n) : n \geq 0\}$, and letting the $P^{\pi,\hat{\pi}}_\mu$-distribution of $M(0)$ be independent of $\{M(n) - M(0) : n \geq 0\}$ and absolutely continuous with respect to m_μ with Radon-Nikodym derivative

$$\frac{c_N}{h_\pi(\mu)} e^{-\sum_{i=1}^{N} \mu_i} \exp(-\mathrm{tr}[(\mathrm{diag}(\pi) - I)M(0)]), \tag{3.1}$$

where c_N is a constant depending only on the dimension N and I is the identity matrix. Recall that $h_\pi(\mu)$ is defined in (2.2). That this defines the density of a probability measure for all π and μ follows immediately from the Harish-Chandra-Itzykson-Zuber formula (e.g., Mehta (2004, App. A.5))

$$\int_U \exp\left(-\mathrm{tr}(\mathrm{diag}(\pi) U \,\mathrm{diag}(\mu) U^*)\right) dU = c_N^{-1} h_\pi(\mu), \tag{3.2}$$

writing dU for normalized Haar measure on the unitary group. Throughout, we abbreviate $P_\mu^{(1,\ldots,1),(0,0,\ldots)}$ by P_μ. Note that the $P_\mu^{\pi,\hat\pi}$-law and the $P^{\pi,\hat\pi}$-law of $\{M(n) : n \geq 0\}$ coincide if $\mu = 0$.

The following theorem specifies the $P_\mu^{\pi,\hat\pi}$-law of $\{\mathrm{sp}(M(n)) : n \geq 0\}$.

Theorem 3.1. *For any $\mu \in W^N$, $\{\mathrm{sp}(M(n)) : n \geq 0\}$ is an inhomogeneous Markov chain on W^N under $P_\mu^{\pi,\hat\pi}$, and it has the $Q_{n-1,n}^{\pi,\hat\pi}$ of Section 2.2 for its one-step transition kernels.*

Proof. Fix some $\mu \in W^N$. The key ingredient in the proof is a change of measure argument. We know from Defosseux (2008) or Forrester and Rains (2006) that Theorem 3.1 holds for the 'standard' case $\pi = (1, \ldots, 1)$, $\hat\pi \equiv 0$.

Writing $P_n^{\pi,\hat\pi}$ and P_n for the distribution of $(M(0), \ldots, M(n))$ under $P_\mu^{\pi,\hat\pi}$ and P_μ respectively, we obtain from Section 2.1 that for $n \geq 0$,

$$\begin{aligned}
&\frac{dP_n^{\pi,\hat\pi}}{dP_n}(M(0), \ldots, M(n)) \\
&\quad= C_{\pi,\hat\pi}(n,N) \frac{c_N}{h_\pi(\mu)} e^{-\sum_{i=1}^N \mu_i} \\
&\qquad \times \exp\left(-\mathrm{tr}((\mathrm{diag}(\pi) - I) M(n)) - \sum_{m=1}^n \hat\pi_m \mathrm{tr}(M(m) - M(m-1))\right),
\end{aligned}$$

where $C_{\pi,\hat\pi}(n,N) = \prod_{i=1}^N \prod_{j=1}^n (\pi_i + \hat\pi_j)$. Let the measure $p_n^{\pi,\hat\pi}$ (and p_n) be the restriction of $P_n^{\pi,\hat\pi}$ (and P_n) to the σ-field generated by $(\mathrm{sp}(M(0)), \ldots, \mathrm{sp}(M(n)))$. Then we obtain for $n \geq 0$,

$$\begin{aligned}
&\frac{dp_n^{\pi,\hat\pi}}{dp_n}(\mathrm{sp}(M(0)), \ldots, \mathrm{sp}(M(n))) \\
&\quad= \mathbb{E}_{P_\mu}\left[\frac{dP_n^{\pi,\hat\pi}}{dP_n}(M(0), \ldots, M(n)) \,\middle|\, \mathrm{sp}(M(0)), \ldots, \mathrm{sp}(M(n))\right],
\end{aligned}$$

where $\mathbb{E}_{P_\mu}$ denotes the expectation operator with respect to P_μ. Since the P_μ-distribution of $(M(0), \ldots, M(n))$ given the spectra is invariant under componentwise conjugation by a unitary matrix U, we have for $\mu \equiv \mu^{(0)} \prec \mu^{(1)} \prec \ldots \prec \mu^{(n)}$,

$$\begin{aligned}
&\mathbb{E}_{P_\mu}\left[\exp\left(-\mathrm{tr}(\mathrm{diag}(\pi) M(n))\right) \,\middle|\, \mathrm{sp}(M(0)) = \mu^{(0)}, \ldots, \mathrm{sp}(M(n)) = \mu^{(n)}\right] \\
&\quad= \int_U \exp\left(-\mathrm{tr}(\mathrm{diag}(\pi) U \,\mathrm{diag}(\mu^{(n)}) U^*)\right) dU \\
&\quad= c_N^{-1} h_\pi(\mu^{(n)}),
\end{aligned}$$

where the second equality is the Harish-Chandra-Itzykson-Zuber formula. From the preceding three displays in conjunction with $\mathrm{tr}(M) = \sum_i \mathrm{sp}(M)_i$, we conclude

that

$$\begin{aligned}
&\frac{dp_n^{\pi,\hat{\pi}}}{dp_n}(\mu,\mu^{(1)},\dots,\mu^{(n)}) \\
&= C_{\pi,\hat{\pi}}(n,N)\frac{h_\pi(\mu^{(n)})}{h_\pi(\mu)}\exp\left(-\sum_{i=1}^N\sum_{r=1}^n \hat{\pi}_r\left[\mu_i^{(r)}-\mu_i^{(r-1)}\right]+\sum_{i=1}^N[\mu_i^{(n)}-\mu_i]\right).
\end{aligned}$$

Since $\mathrm{sp}(M(\cdot))$ is a Markov chain with transition kernel Q under P_μ, we have

$$\begin{aligned}
&P_\mu^{\pi,\hat{\pi}}(\mathrm{sp}(M(1))\in d\mu^{(1)},\dots,\mathrm{sp}(M(n))\in d\mu^{(n)}) \\
&= \frac{dp_n^{\pi,\hat{\pi}}}{dp_n}(\mu,\mu^{(1)},\dots,\mu^{(n)})P_\mu(\mathrm{sp}(M(1))\in d\mu^{(1)},\dots,\mathrm{sp}(M(n))\in d\mu^{(n)}) \\
&= \frac{dp_n^{\pi,\hat{\pi}}}{dp_n}(\mu,\mu^{(1)},\dots,\mu^{(n)})Q(\mu,d\mu^{(1)})\cdots Q(\mu^{(n-1)},d\mu^{(n)}) \\
&= Q_{0,1}^{\pi,\hat{\pi}}(\mu,d\mu^{(1)})Q_{1,2}^{\pi,\hat{\pi}}(\mu^{(1)},d\mu^{(2)})\cdots Q_{n-1,n}^{\pi,\hat{\pi}}(\mu^{(n-1)},d\mu^{(n)}),
\end{aligned}$$

the last equality being a consequence of the definition of $Q_{k-1,k}^{\pi,\hat{\pi}}$ and the expression for $dp_n^{\pi,\hat{\pi}}/dp_n$. □

4. Robinson-Schensted-Knuth and the proof of Theorem 1.1

This section explains the connection between the infinite array $\{W_{ij}\}$ of the introduction and the Markov kernels $Q_{n-1,n}^{\pi,\hat{\pi}}$. In conjunction with Theorem 3.1, these connections allow us to prove Theorem 1.1.

The RSK algorithm. The results in this section rely on a combinatorial mechanism known as the Robinson-Schensted-Knuth (RSK) algorithm. This algorithm generates from a $p\times q$ matrix with nonnegative entries a triangular array $\mathbf{x}=\{x_i^j : 1\le j\le p, 1\le i\le j\}$ called a Gelfand-Tsetlin (GT) pattern. A GT pattern with p levels $x^1,\dots,x^p$ is an array for which the coordinates satisfy the inequalities

$$x_k^k \le x_{k-1}^{k-1} \le x_{k-1}^k \le x_{k-2}^{k-1} \le \dots \le x_2^k \le x_1^{k-1} \le x_1^k$$

for $k=2,\dots,p$. If the elements of the matrix are integers, then a GT pattern can be identified with a so-called semistandard Young tableau, and the bottom row $x^p=\{x_i^p; 1\le i\le p\}$ of the GT pattern corresponds to the shape of the Young tableau. We write $\mathbf{K}_p$ for the space of all GT patterns $\mathbf{x}$ with p levels.

By applying the RSK algorithm with row insertion to an infinite array $\{\xi_{ij} : 1\le i\le N, 1\le j\le n\}$ for $n=1,2,\dots$, we obtain a sequence of GT patterns $\mathbf{x}(1),\mathbf{x}(2),\dots$. It follows from properties of RSK that

$$x_1^N(n) = \max_{P\in\Pi(N,n)}\sum_{(ij)\in P}\xi_{ij}, \tag{4.1}$$

where $\Pi(N,n)$ is the set of up-right paths from $(1,1)$ to (N,n) as before. Details can be found in, e.g., Johansson (2000) or Dieker and Warren (2008, case A).

Greene's theorem generalizes (4.1), and gives similar expressions for each component of the pattern $x_i^j(n)$, see for instance Chapter 3 of Fulton (1997) or Equation (16) in Doumerc (2003). As a consequence of these, we can consider the RSK algorithm for real-valued ξ_{ij} and each $\mathbf{x}(n)$ is then a continuous function of the input data.

We remark that the RSK algorithm can also be started from a given initial GT pattern $\mathbf{x}(0)$. If RSK is started from the null pattern, it reduces to the standard algorithm and we set $\mathbf{x}(0) = 0$.

The bijective property of RSK. RSK has a bijective property which has important probabilistic consequences for the sequence of GT patterns constructed from specially chosen random infinite arrays. Indeed, suppose that $\{\xi_{ij} : 1 \le i \le N, j \ge 1\}$ is a family of independent random variables with ξ_{ij} having a geometric distribution on $\mathbb{Z}_+$ with parameter $a_i b_j$, where $\{a_i : 1 \le i \le N\}$ and $\{b_j : j \ge 1\}$ are two sequences taking values in $(0, 1]$. Write $\{\mathbf{X}(n) : n \ge 0\}$ for the sequence of GT patterns constructed from ξ.

Using the bijective property of RSK it can be verified that the bottom rows $\{X^N(n) : n \ge 0\}$ of the GT patterns evolve as an inhomogeneous Markov chain with transition probabilities

$$P_{n-1,n}(x, x') = \prod_{i=1}^{N}(1 - a_i b_n)\frac{s_{x'}(a)}{s_x(a)} b_n^{\sum_{i=1}^{N}(x'_i - x_i)} 1_{\{0 \le x \prec x'\}}, \tag{4.2}$$

where $s_\lambda(a)$ is the Schur polynomial corresponding to a partition λ:

$$s_\lambda(a) = \sum_{\mathbf{x} \in \mathbf{K}_N : x^N = \lambda} a^{\mathbf{x}},$$

with the weight $a^{\mathbf{x}}$ of a GT pattern $\mathbf{x}$ being defined as

$$a^{\mathbf{x}} = a_1^{x_1^1} \prod_{k=2}^{N} a_k^{\sum x_i^k - \sum x_i^{k-1}}.$$

This is proved in O'Connell (2003) in the special case with $b_j = 1$ for all j, and the argument extends straightforwardly; see also Forrester and Nagao (2008).

Non-null initial GT patterns generally do *not* give rise to Markovian bottom-row processes. Still, the inhomogeneous Markov chain of bottom rows can be constructed starting from a given initial partition λ with at most N parts by choosing $\mathbf{X}(0)$ suitably from the space of a GT patterns with bottom row λ: $\mathbf{X}(0)$ should be independent of the family $\{\xi_{ij}\}$ with probability mass function

$$p(\mathbf{x}) = \frac{a^{\mathbf{x}}}{s_\lambda(a)}.$$

Exponentially distributed input data. We now consider the sequence of GT patterns $\{\mathbf{X}_L(n) : n \ge 0\}$ arising from setting $a_i = 1 - \pi_i/L$ and $b_j = 1 - \hat{\pi}_j/L$ in the above setup, and we study the regime $L \to \infty$ after rescaling suitably. In the regime $L \to \infty$, the input variables $\{\xi_{ij}/L\}$ (jointly) converge in distribution to independent exponential random variables, the variable corresponding to ξ_{ij}/L having parameter $\pi_i + \hat{\pi}_j$. Thus, the law of the input array ξ converges weakly to the $P^{\pi,\hat{\pi}}$-law of the array $\{W_{ij} : 1 \le i \le N, j \ge 1\}$ from the introduction. Refer to Doumerc (2003) and Johansson (2000) for related results on this regime.

By the aforementioned continuity of the RSK algorithm and the continuous-mapping theorem, $\{\mathbf{X}_L(n)/L : n \ge 0\}$ converges in distribution to a process $\{\mathbf{Z}(n) :$

$n \geq 0\}$ taking values in GT patterns with N levels. As a consequence of the above results in a discrete-space setting, we get from (4.1) that

$$Z_1^N(n) = \max_{P \in \Pi(N,n)} \sum_{(ij) \in P} W_{ij}.$$

Moreover, the process of bottom rows $\{Z^N(n) : n \geq 0\}$ is an inhomogeneous Markov chain for which its transition mechanism can be found by letting $L \to \infty$ in (4.2):

Lemma 4.1. *Under $P^{\pi,\hat{\pi}}$, the process $\{Z^N(n) : n \geq 0\}$ is an inhomogeneous Markov chain on W^N, and it has the $Q^{\pi,\hat{\pi}}_{n-1,n}$ of Section 2.2 for its one-step transition kernels.*

A similar result can be obtained given a non-null initial bottom row $\mu \in W^N$. In case the components of μ are distinct, the distribution of the initial pattern $\mathbf{Z}(0)$ should then be absolutely continuous with respect to Lebesgue measure on $\{\mathbf{z} \in \mathbf{K}_N : z^N = \mu\}$ with density

$$\frac{\Delta(\pi)}{\det\{e^{-\pi_i \mu_j}\}} c^{\mathbf{z}},$$

where $c = (e^{-\pi_1}, \ldots, e^{-\pi_N})$.

Proof of Theorem 1.1. We now have all ingredients to prove Theorem 1.1. We already noted that $Z_1^N(n)$ equals $Y(N,n)$. Thus, for any strictly positive vector π and any nonnegative sequence $\hat{\pi}$, $\{Y(N,n) : n \geq 1\}$ has the same $P^{\pi,\hat{\pi}}$-distribution as $\{Z_1(n) : n \geq 1\}$. In view of Theorem 3.1 and Lemma 4.1, in turn this has the same $P^{\pi,\hat{\pi}}$-distribution as the largest-eigenvalue process $\{\mathrm{sp}(M(n))_1 : n \geq 1\}$. This proves Theorem 1.1.

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